NANOSYSTEMS: PHYSICS, CHEMISTRY, MATHEMATICS, 2021, 12 (2), P. 135–141

Non-compact perturbations of the spectrum of multipliers given with functions

R. R. Kucharov[1], R. R. Khamraeva[1,2]

[1]National University of Uzbekistan, 100174, Tashkent, Uzbekistan
[2]Westminster International University in Tashkent, 100010, 12, Istiqbol str., Tashkent, Uzbekistan

ramz3364647@yahoo.com, r.khamraeva@wiut.uz

DOI 10.17586/2220-8054-2021-12-2-135-141

The change in the spectrum of the multipliers $H_0f(x,y) = x^\alpha + y^\beta f(x,y)$ and $H_0f(x,y) = x^\alpha y^\beta f(x,y)$ for perturbation with partial integral operators in the spaces $L_2[0,1]^2$ is studied. Precise description of the essential spectrum and the existence of simple eigenvalue is received. We prove that the number of eigenvalues located below the lower edge of the essential spectrum in the model is finite.

Keywords: essential spectrum, discrete spectrum, lower bound of the essential spectrum, partial integral operator.

Received: 25 January 2021

Revised: 10 March 2021

Final revision: 12 March 2021

1. Introduction

The first results on the finiteness of the discrete spectrum of N – particle Hamiltonians with $N > 2$ were obtained by Uchiyama in 1969 [1–3]. He found sufficient conditions for the finiteness of the number of discrete eigenvalues for energy operators in the space $L_2(\mathbb{R}^6)$ for the system of two identically charged particles in the field of a fixed center with or without an external electromagnetic field. In 1971, Zhislin proved the finiteness of the discrete spectrum for energy operators in symmetry spaces of negative atomic ions with nuclei of any mass and of molecules with infinitely heavy nuclei under the assumption that the total charge of the system is less than -1 [4].

Let Ω_1 and Ω_2 be closed bounded sets in $\mathbb{R}^{\nu_1}$ and $\mathbb{R}^{\nu_2}$, respectively. In the space $L_p(\Omega_1 \times \Omega_2),\ p \geq 1$ partially integral operator (PIO) T of the Fredholm type in general is given by the equality [5]:

$$T = T_0 + T_1 + T_2 + K, \tag{1}$$

where the operators T_0, T_1, T_2, K have the following view:

$$T_0f(x,y) = k_0(x,y)f(x,y), \quad T_1f(x,y) = \int_{\Omega_1} k_1(x,s,y)f(s,y)d\mu_1(s),$$

$$T_2f(x,y) = \int_{\Omega_2} k_2(x,t,y)f(x,t)d\mu_2(t), \quad Kf(x,y) = \int_{\Omega_1}\int_{\Omega_2} k(x,y;s,t)f(s,t)d\mu_1(s)d\mu_2(t).$$

Here, the functions k_0, k_1, k_2, and k are given measurable functions in the concept of Lebesque on $\Omega_1 \times \Omega_2$, $\Omega_1^2 \times \Omega_2$, $\Omega_1 \times \Omega_2^2$ and $(\Omega_1 \times \Omega_2)^2$, respectively, and integration of functions is understood in the concept of Lebesgue, where $\mu_k(\cdot)$ – Lebesque measure on $\Omega_k,\ k = 1,2$.

In the Hilbert space $L_2(\Omega \times \Omega)$, where $\Omega = [a,b]^\nu$, consider the following model operator:

$$H = H_0 - (\gamma T_1 + \mu T_2),\ \gamma > 0,\ \mu > 0. \tag{2}$$

Here, the actions of the operators H_0, T_1 and T_2 are determined by formulas:

$$H_0f(x,y) = k_0(x,y)f(x,y),$$

$$T_1f(x,y) = \int_\Omega \varphi_1(x)\varphi_1(s)f(s,y)ds, \quad T_2f(x,y) = \int_\Omega \varphi_2(y)\varphi_2(t)f(x,t)dt,$$

where $k_0(x,y)$ is a nonnegative continuous function on $\Omega \times \Omega$, $\varphi_j(\cdot)$ is a continuous function on Ω and

$$\int_\Omega \varphi_j^2(t)dt = 1,\ j = 1,2.$$

Via $\rho(\cdot), \sigma(\cdot), \sigma_{ess}(\cdot)$ and $\sigma_{disc}(\cdot)$ denote, respectively, the resolvent set, spectrum, essential spectrum and discrete spectrum self-adjoint operators [6].

In [7], sufficient conditions for finiteness and infinity were obtained in the discrete spectrum for $\sigma_{ess}(H) = \sigma(H_0)$. In work [8] proved the existence of the Efimov effect in model (2) for given $k_0(x, y)$. In [9], the essential spectrum and the number eigenvalues below the lower bound of the essential spectrum in model (2), when the function $k_0(x, y)$ has the form: $k_0(x, y) = u(x)u(y)$, where $u(x)$ is a nonnegative continuous function on $\Omega = \Omega_1 = \Omega_2$ and $\int\limits_{\Omega} \frac{dx}{u(x)} < \infty$. In [10] studied the existence of an infinite number of eigenvalues (the existence of Efimov's effect) for a selfadjoint partial integral operators.

2. The lower boundary of the essential spectrum of *V*

Consider the multiplier:

$$V_0 f(x, y) = (x^\alpha + y^\beta) f(x, y), \;\; \alpha > 0, \;\; \beta > 0.$$

Let us define a partially integral operator (PIO) V:

$$V = V_0 - \gamma(T_1 + T_2), \;\; \gamma > 0, \tag{3}$$

where:

$$T_1 f(x, y) = \int\limits_0^1 f(s, y) ds, \quad T_2 f(x, y) = \int\limits_0^1 f(x, t) dt, \;\; f \in L_2[0, 1]^2.$$

In the space $L_2[0, 1]$ we define the operators H_1 and H_2 in Friedrichs models:

$$H_1 \varphi(x) = x^\alpha \varphi(x) - \gamma \int\limits_0^1 \varphi(s) ds, \quad H_2 \psi(y) = y^\beta \psi(y) - \gamma \int\limits_0^1 \psi(t) dt.$$

Lemma 1. [11] The number $\lambda \in \mathbb{R} \backslash [0, 1]$ is the eigenvalue of the operator H_1 (of the operator H_2) if and only if $\Delta_1(\lambda) = 0$ ($\Delta_2(\lambda) = 0$), where:

$$\Delta_1(\lambda) = 1 - \gamma \int\limits_0^1 \frac{ds}{s^\alpha - \lambda}, \quad \Delta_2(\lambda) = 1 - \gamma \int\limits_0^1 \frac{ds}{s^\beta - \lambda}.$$

Lemma 2.

A)

$$\lim_{\lambda \to 0-} \Delta_1(\lambda) = \begin{cases} 1 - \dfrac{\gamma}{1-\alpha}, & \text{if } 0 < \alpha < 1; \\ -\infty, & \text{if } \alpha \geq 1, \end{cases}$$

B)

$$\lim_{\lambda \to 0-} \Delta_2(\lambda) = \begin{cases} 1 - \dfrac{\gamma}{1-\beta}, & \text{if } 0 < \beta < 1; \\ -\infty, & \text{if } \beta \geq 1. \end{cases}$$

Proof. First, we prove the statement A.

a) Let $0 < \alpha < 1$. Consider an arbitrary increasing sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ of negative numbers approaching to zero, i.e $\lambda_n \leq \lambda_{n+1} < 0$ and $\lim\limits_{n \to \infty} \lambda_n = 0$. Then:

$$0 < \frac{1}{s^\alpha - \lambda_n} \leq \frac{1}{s^\alpha - \lambda_{n+1}}, \;\; n \in \mathbb{N}$$

and

$$\frac{1}{s^\alpha - \lambda_n} \leq \frac{1}{s^\alpha} \;\; \text{for almost all} \;\; s \in [0, 1].$$

The function $h_0(s) = \frac{1}{s^\alpha}$ is integrable by $[0, 1]$ in the concept of Lebesgue and:

$$\int\limits_0^1 h_0(s) ds = \frac{1}{1-\alpha}.$$

Hence, due to Lebesgue theorem on limited transition under the sign of the Lebesgue integral it follows that:

$$\lim_{\lambda\to 0-}\int\limits_0^1 \frac{ds}{s^\alpha-\lambda}=\frac{1}{1-\alpha}.$$

Thus, we have:

$$\lim_{\lambda\to 0-}\Delta_1(\lambda)=1-\frac{\gamma}{1-\alpha}.$$

b) Let $\alpha\geq 1$. Suppose that $\alpha=1$. Then:

$$\lim_{\lambda\to 0-}\Delta_1(\lambda)=1-\gamma\lim_{\lambda\to 0-}\int\limits_0^1\frac{ds}{s-\lambda}=1-\gamma\lim_{\lambda\to 0-}\ln\left(1-\frac{1}{-\lambda}\right)=-\infty.$$

If $\alpha>1$, then we have:

$$\frac{1}{s^\alpha-\lambda}\geq\frac{1}{s-\lambda},\ \ s\in[0,1].$$

Hence:

$$\lim_{\lambda\to 0-}\int\limits_0^1\frac{ds}{s^\alpha-\lambda}\geq\lim_{\lambda\to 0-}\int\limits_0^1\frac{ds}{s-\lambda}=+\infty,$$

i.e.

$$\lim_{\lambda\to 0-}\Delta_1(\lambda)=-\infty.$$

Proposition 1. I) Let $0<\alpha<1$ $(0<\beta<1)$. Then:

a) if $\alpha+\gamma\leq 1$ $(\beta+\gamma\leq 1)$, then the operator H_1 (operator H_2) outside the essential spectrum has the only eigenvalue of the operator;

b) if $\alpha+\gamma>1$ $(\beta+\gamma>1)$, then the operator H_1 (operator H_2) outside the essential spectrum has the only eigenvalue of the operator ξ_1 (the eigenvalue value ξ_2), while ξ_k is a simple proper value H_k $\xi_k<0$, $k=1,2$.

II) Let $\alpha\geq 1$ $(\beta\geq 1)$. Then the operator H_1 (operator H_2) outside the essential spectrum has a unique eigenvalue ξ_1 (eigenvalue ξ_2), for this ξ_k is a simple eigenvalue of the operator H_k and $\xi_k<0$, $k=1,2$.

Proof. It is easy to note that the function $\Delta_1(\lambda)$ by $(-\infty,0)$ is strictly decreasing and $\Delta_1(\lambda)>0$ to $(1,\infty)$, thus, the operator H_1 on the set $(1,\infty)$ has no eigenvalue.

Let $0<\alpha<1$. By Lemma 2 and monotonicity, the function $\Delta_1(\lambda)$ to $(-\infty,0)$ states a and b, since:

$$\mathrm{Ran}(\Delta_1)=\left(1-\frac{\gamma}{1-\alpha},1\right).$$

Let $\alpha\geq 1$. Then from Lemma 2 we obtain: $\mathrm{Ran}(\Delta_1)=(-\infty,1)$. Due to monotonicity functions $\Delta_1(\lambda)$ $(-\infty,0)$ equation $\Delta_1(\lambda)=0$ $(-\infty,0)$ has a unique solution $\xi_1<0$ ξ_1 is a simple eigenvalue operator H_1.

Proposition 1 for the operator H_2 is proved similarly.

Theorem 1. Let $0<\alpha<1$ and $0<\beta<1$. Then:

a) if $\alpha+\gamma\leq 1$ and $\alpha+\beta\leq 1$, then:

$$\sigma(V)=\sigma_{ess}(V)=\sigma(V_0)=[0,2];$$

b) if $\alpha+\gamma>1$ and $\alpha+\beta\leq 1$, then

$$\sigma(V)=\sigma_{ess}(V)=\sigma(V_0)\cup[\xi_1,1+\xi_1],$$

where ξ_1 – negative eigenvalue of operator H_1;

c) if $\alpha+\gamma\leq 1$ and $\alpha+\beta>1$, then:

$$\sigma_{ess}(V)=\sigma(V_0)\cup[\xi_2,1+\xi_2],$$

where ξ_2 is a negative eigenvalue of operator H_2;

d) if $\alpha+\gamma>1$ and $\alpha+\beta>1$, then:

$$\sigma_{ess}(V)=\sigma(V_0)\cup[\xi_1,1+\xi_1]\cup[\xi_2,1+\xi_2]\ \text{ and }\ \sigma_{disc}(V)=\{\omega_0\},$$

where $\omega_0=\xi_1+\xi_2$ and ω_0 is a simple eigenvalue of operator V.

Proof. It is easy to note that the operator V will be unitarily equivalent to the operator $H_1\otimes E+E\otimes H_2$ (see [12]). Then $\sigma(V)=\sigma(H_1)+\sigma(H_2)$ and for of multiplicity $n_V(\omega)$ eigenvalue $\omega\in\sigma(V)\setminus\sigma_{ess}(V)$ of the operator V the following equality takes place:

$$n_V(\omega)=\sum_{\substack{p+q=\omega\\(p,q)\in\sigma(H_1)\times\sigma(H_2)}} n_{H_1}(p)\cdot n_{H_2}(q),$$

where $n_{H_1}(p)$ and $n_{H_2}(q)$ – multiplicity of the eigenvalues p and q of the operators H_1 and H_2, respectively. This and Proposition 1 imply the proof the theorem.

Theorem 2. Let $\alpha\geq 1$ and $0<\beta<1$. Then:

a) if $\beta+\gamma\leq 1$, then:

$$\sigma(V)=\sigma_{ess}(V)=\sigma(V_0)\cup[\xi_1,1+\xi_1];$$

b) if $\beta+\gamma>1$, then:

$$\sigma_{ess}(V)=\sigma(V_0)\cup[\xi_1,1+\xi_1]\cup[\xi_2,1+\xi_2]\ \text{and}\ \sigma_{disc}(V)=\{\omega_0\},$$

where $\omega_0=\xi_1+\xi_2$ and ω_0 is a simple eigenvalue of the operator V.

Theorem 3. Let $\alpha\geq 1$ and $\beta\geq 1$. Then:

$$\sigma_{ess}(V)=\sigma(V_0)\cup[\xi_1,1+\xi_1]\cup[\xi_2,1+\xi_2]\ \text{and}\ \sigma_{disc}(V)=\{\omega_0\},$$

where $\omega_0=\xi_1+\xi_2$ è ω_0 – is a simple eigenvalue of the operator V.

Corollary 1. Let $0<\alpha<1$ and $0<\beta<1$. Then:

$$E_{\min}(V)=\begin{cases}0, & \text{if } \alpha+\gamma\leq 1 \text{ and } \beta+\gamma\leq 1,\\ \xi_1, & \text{if } \alpha+\gamma>1 \text{ and } \beta+\gamma\leq 1,\\ \xi_2, & \text{if } \alpha+\gamma\leq 1 \text{ and } \beta+\gamma\geq 1,\\ \min\{\xi_1,\xi_2\}, & \text{if } \alpha+\gamma>1 \text{ and } \beta+\gamma>1.\end{cases}$$

3. Discrete spectrum of partial integral operators

Let's define the multiplier H_0:

$$H_0f(x,y)=x^\alpha y^\beta f(x,y),\quad \alpha>0,\ \beta>0,$$

and the operators T_1, T_2:

$$T_1f(x,y)=\int_0^1 f(s,y)ds,\quad T_2f(x,y)=\int_0^1 f(x,t)dt.$$

Let us define a self-conjugate PIO H:

$$H=H_0-\gamma(T_1+T_2),\ \gamma>0.$$

We have $\sigma(H_0)=[0,1]$. For each $\lambda\in\mathbb{R}\setminus[0,1]$ define the function $\Delta_1(y;\lambda)$ on [0,1] ($\Delta_2(x;\lambda)$ on $[0,1]$) by formula:

$$\Delta_1(y;\lambda)=1-\gamma\int_0^1\frac{ds}{s^\alpha y^\beta-\lambda},\quad \Delta_2(x;\lambda)=1-\gamma\int_0^1\frac{ds}{x^\alpha s^\beta-\lambda}.$$

In the space $L_2[0,1]$ we define the family $\{H_1(t)\}_{t\in[0,1]}$ of the self-adjoint operators in the Friedrichs' model:

$$H_1(t)\varphi(x)=t^\beta x^\alpha\varphi(x)-\gamma\int_0^1\varphi(s)ds.$$

Similarly, in the space $L_2[0,1]$ we define the family $\{H_2(t)\}_{t\in[0,1]}$:

$$H_2(t)\psi(y)=t^\alpha y^\beta\psi(y)-\gamma\int_0^1\psi(s)ds.$$

Lemma 3. Function:

$$\pi_j(t)=\inf_{\|\varphi\|=1}(H_j(t)\varphi,\varphi),\ t\in[0,1]\ (j=1,2) \tag{3}$$

is non-positive, continuous and increasing on $[0,1]$.

Proof. In work [9], there is a proof of the continuity and non-positivity of the function $\pi_j(t)$ on $[0,1]$. We will show the monotonicity of the function $\pi_j(t)$ on $[0,1]$. We define the family of the $\{H_0(t)\}_{t\in[0,1]}$ multipliers:

$$H_0(t)\varphi(x) = x^\alpha t^\beta \varphi(x),\ \ \varphi \in L_2[0,1].$$

Then it follows from $t_1 \le t_2,\ \ t_1, t_2 \in [0,1]$ that:

$$H_0(t_1) \le H_0(t_2).$$

Therefore, we have:

$$\pi_1(t_1) = \inf_{\|\varphi\|=1}(H_1(t_1)\varphi,\varphi) = \inf_{\|\varphi\|=1}[(H_0(t_1)\varphi,\varphi) - \gamma(K_1\varphi,\varphi)] \le$$

$$\inf_{\|\varphi\|=1}(H_1(t_2)\varphi,\varphi) = \inf_{\|\varphi\|=1}[(H_0(t_2)\varphi,\varphi) - \gamma(K_1\varphi,\varphi)] = \pi_1(t_2),$$

Where:

$$K_1\varphi(x) = \int_0^1 \varphi(s)ds.$$

This means that the function $\pi_1(t)$ is increasing on the set $[0,1]$.

Obviously, for each $y \in [0,1]$ the function $\Delta_1(\lambda) = \Delta_1(y;\lambda)$ is strictly decreasing on $(-\infty, 0)$. Therefore, for each $y \in [0,1]$ there exists finite or infinite limit $\lim\limits_{\lambda\to 0-} \Delta_1(y;\lambda)$. Moreover, there is:

Lemma 4. a) if $0 < \alpha < 1$, then for each $y \in (0,1]$:

$$\lim_{\lambda\to 0-} \Delta_1(y;\lambda) = 1 - \frac{\gamma}{1-\alpha}\cdot\frac{1}{y^\beta};$$

b) if $\alpha \ge 1$, then for each $y \in (0,1]$:

$$\lim_{\lambda\to 0-} \Delta_1(y;\lambda) = -\infty.$$

Proof. a) Let $0 < \alpha < 1$. Then, for $y \in (0,1]$ we get

$$\frac{1}{s^\alpha y^\beta - \lambda} \le h_0(s,y) = \frac{1}{s^\alpha y^\beta}$$

and for any ascending sequence $\{\lambda_n\}$ negative numbers decreasing to zero we have:

$$\frac{1}{s^\alpha y^\beta - \lambda_n} \le \frac{1}{s^\alpha y^\beta - \lambda_{n+1}},\ \ n \in \mathbb{N}.$$

On the other hand, for each $y \in (0,1]$ there exists a Lebesgue integral from function $h_1(s,y)$ on $s \in [0,1]$:

$$\int_0^1 h_0(s,y)ds = \frac{1}{1-\alpha}\cdot\frac{1}{y^\beta}.$$

Then, by Lebesgue's theorem on the limited transition under the sign of the Lebesgue integral, we obtain:

$$\lim_{\lambda\to 0-} \Delta_1(y;\lambda) = 1 - \frac{\gamma}{1-\alpha}\cdot\frac{1}{y^\beta},\ \ y \in (0,1];$$

b) Let $\alpha \ge 1$ and assume that $\alpha = 1$. It is obvious that for $y = 0$ we have: $\lim\limits_{\lambda\to 0-} \Delta_1(y;\lambda) = -\infty$. For each $y \in (0,1]$ we have:

$$\int_0^1 \frac{ds}{s^\alpha y^\beta - \lambda} = \int_0^1 \frac{ds}{sy^\beta - \lambda} = \frac{1}{y^\beta}\ln\left(1 - \frac{y^\beta}{\lambda}\right).$$

Therefore for $y \in (0,1]$ we get:

$$\lim_{\lambda\to 0-} \Delta_1(y;\lambda) = 1 - \frac{\gamma}{y^\beta}\lim_{\lambda\to 0-}\ln\left(1 - \frac{y^\beta}{\lambda}\right) = -\infty;$$

Suppose that $\alpha > 1$. Then from inequality:

$$\int_0^1 \frac{ds}{s^\alpha y^\beta - \lambda} \geq \int_0^1 \frac{ds}{sy^\beta - \lambda}, \ \ y \in [0,1]$$

we get that

$$\lim_{\lambda \to 0-} \int_0^1 \frac{ds}{s^\alpha y^\beta - \lambda} = +\infty, \ \ y \in [0,1]$$

and accordingly, $\lim\limits_{\lambda \to 0-} \Delta_1(y;\lambda) = -\infty$.

Obviously, the function:

$$h_1(y) = \lim_{\lambda \to 0-} \Delta_1(y;\lambda) = 1 - \frac{\gamma}{1-\alpha} \cdot \frac{1}{y^\beta}$$

increases by $(0,1]$ from $-\infty$ to $h_1^{\max} = h_1(1) = 1 - \dfrac{\gamma}{1-\alpha}$.

We put:

$$\pi_j^{\max} = \max_{t \in [0,1]} \pi_j(t), \ \ j = 1,2.$$

Then $\pi_j^{\max} = \pi_j(1)$.

Lemma 5. Let $0 < \alpha < 1$ $(0 < \beta < 1)$. Then:

a) if $\gamma + \alpha \leq 1$ $(\gamma + \beta \leq 1)$, then $\pi_1^{\max} = 0$ $(\pi_2^{\max} = 0)$;

b) if $\gamma + \alpha > 1$ $(\gamma + \beta > 1)$, then $\pi_1^{\max} < 0$ $(\pi_1^{\max} < 0)$.

Proof. Let $0 < \alpha < 1$. a) Assume that: $\gamma + \alpha = 1$. We have:

$$h_1^{\max} = h_1(1) = \lim_{\lambda \to 0-} \Delta_1(1;\lambda) = 1 - \frac{\gamma}{1-\alpha} = 0.$$

Hence, taking into account the monotonicity of the function $\Delta_1(1;\lambda)$ by $\lambda < 0$ we get that $\Delta_1(1;\lambda) > 0$ for any $\lambda < 0$. Then, according to Proposition 1, the operator $H_1(1)$ has no eigenvalue below the bottom edge the essential spectrum of the operator $H_1(1)$. By the minmax principle and from equality (3) we obtain that $\pi_1^{\max} = \pi_1(1) = E_{\min}(H_1(1)) = 0$.

If $\gamma + \alpha < 1$. Then $h_1^{\max} = h_1(1) > 0$. On the other hand $\Delta_1(1;\lambda) > h_1^{\max}$. Then according to the proposition 1, the operator $H_1(1)$ has no negative eigenvalue value. It follows that $\pi_1^{\max} = 0$.

b) Let $\gamma + \alpha > 1$. Then:

$$h_1(y) \leq h_1^{\max} = 1 - \frac{\gamma}{1-\alpha} < 0.$$

Therefore, for for each $y \in (0,1]$ we have $h_1(y) = \lim\limits_{\lambda \to 0-} \Delta_1(y;\lambda) < 0$. Hence, since the function $\Delta_1(y;\lambda)$ is monotonic with respect to $\lambda < 0$ implies the existence of a unique number $\lambda_0 = \lambda_0(y) < 0$ (for each $y \in (0,1]$) such that $\Delta_1(y;\lambda_0(y)) = 0$. For $y = 0$ we have $\Delta_1(0;-\gamma) = 0$, i.e $\lambda_0 = \lambda_0(0) = -\gamma$ is a solution to the equation $\Delta_1(0;\lambda) = 0$. Due to minmax principle [13] solution of $\lambda_0(y)$ equation $\Delta_1(y;\lambda) = 0$ is defined using continuous function $\pi_1(t)$, i.e. $\lambda_0(t) = \pi_1(t)$, $t \in [0,1]$. However $\lambda_0(y) < 0$, $y \in [0,1]$. Therefore $\pi_1(1) = \pi_1^{\max} < 0$.

Lemma 6. Let $\alpha \geq 1$ $(\beta \geq 1)$. Then $\pi_1^{\max} < 0$ $(\pi_2^{\max} < 0)$.

Proof. For $y = 1$ we get:

$$\Delta_1(\lambda) = \Delta_1(1;\lambda) = 1 - \int_0^1 \frac{ds}{s^\alpha - \lambda}.$$

We have:

$$\lim_{\lambda \to -\infty} \Delta_1(\lambda) = 1 \ \text{ and } \ \lim_{\lambda \to 0-} \Delta_1(\lambda) = -\infty.$$

Then, due to the monotonicity of the function $\Delta_1(\lambda)$ for $\lambda < 0$ we obtain the existence of unique number $\lambda_0 < 0$, such as $\Delta_1(\lambda_0) = 0$. Therefore, $\lambda_0 = \pi_1^{\max} < 0$.

By the theorem 3.3 [14] and Lemma 3,4 or the essential spectrum operator H we obtain the following statement.

Theorem 4. For the essential spectrum of the operator H there is a place for equality:

$$\sigma_{ess}(H) = [-\gamma, \gamma_0] \cup [0,1],$$

where $\gamma_0 = \max\{\pi_1^{\max}, \pi_2^{\max}\}$.

From the positivity of the operators H_0, T_1 and T_2 for the operator H we have:

$$\sigma(H) \subset (-\infty, 1],$$

i.e. above the upper edge of the essential spectrum $\sigma_{ess}(H)$ of the operator H eigenvalues are missing. Then, by the theorem 4 the discrete spectrum of the operator H lies in the set of negative numbers.

We put:

$$\xi_0 = \frac{1}{(1+\alpha)(1+\beta)}.$$

Theorem 5. If $\gamma > \xi_0$, then the operator H has the negative eigenvalue, lying to the left of the bottom edge of the essential spectrum.

Proof. Assume the conditions $\gamma > \xi_0$. Put $f_0(x, y) = 1$. Then $\|f_0\| = 1$ and

$$(Hf_0, f_0) = (H_0 f_0, f_0) - \gamma((T_1 f_0, f_0) + (T_2 f_0, f_0)) = \xi_0 - 2\gamma.$$

Then, by the theorem 4 we have $E_{\min}(H) = -\gamma$ and from $\gamma > \xi_0$ we get that

$$\lambda_0 = (Hf_0, f_0) < -\gamma = E_{\min}(H).$$

Hereof and according to the minmax principle we get that $\lambda_0 \in \sigma_{disc}(H)$, i.e. $\lambda_0 = \xi_0 - 2\gamma$ – is the eigenvalue of the operator H.

Corollary 2. Number of eigenvalues of the operator H is at most one and for $\gamma > \xi_0$ the discrete spectrum of the operator H is not empty.

References

[1] Uchiyama J. Finiteness of the Number of Discrete Eigenvalues of the Schrodinger Operator for a Three Particle System. 1969, *Publ. Res. Inst. Math. Sci.*, **5** (1), P. 51–63.

[2] Uchiyama J. Corrections to "Finiteness of the Number of Discrete Eigenvalues of the Schrodinger Operator for a Three Particle System". *Publ. Res. Inst. Math. Sci.*, 1970, **6** (1), P. 189–192.

[3] Uchiyama J. Finiteness of the Number of Discrete Eigenvalues of the Schrodinger Operator for a Three Particle System. *Publ. Res. Inst. Math. Sci.*, 1970, **6** (1), P. 193–200.

[4] Zhislin G.M. On the finiteness of the discrete spectrum of the energy operator of negative atomic and molecular ions. *Theor. Math. Phys.*, 1971, **7**, P. 571–578.

[5] Appell J., Frolova E.V., Kalitvin A.S., Zabrejko P.P. Partial integral operators on $C([a, b] \times [c, d])$. *Integral Equ. Oper. theory*, 1997, **27**, P. 125–140.

[6] Faddeev L.D. On a model of Friedrichs in the theory of perturbations of the continuous spectrum. *Trudy Mat. Inst. Steklov*, 1964, **73**, 292 [in Russian].

[7] Albeverio S., Lakaev S.N., Muminov Z.I. On the number of eigenvalues of a model operator associated to a system of three-particles on lattices. *Russ. J. Math. Phys.*, 2007, **14** (4), P. 377—387.

[8] Rasulov T.Kh. Asymptotics of the discrete spectrum of a model operator assotiated with a system of three particles on a lattice. *Theor. and Math. Phys.*, 2010, **163** (1), P. 429–437.

[9] Eshkabilov Yu.Kh., Kucharov R.R. Essential and discrete spectra of the three-particle Schrodinger operator on a lattice. *Theor. Math. Phys.*, 2012, **170** (3), P. 341-–353.

[10] Eshkabilov Yu.Kh., Kucharov R.R. Efimov's effect for partial integral operators of Fredholm type. *Nanosystems: Physics, Chemistry, Mathematics*, 2013, **4** (4), P. 529–537.

[11] Eshkabilov Yu.Kh. On infinity of the discrete spectrum of operators in the Friedrichs model. *Siberian Adv. Math.*, 2012, **22** (1).

[12] Reed M., Simon B. *Methods of Modern Mathematical Physics, Vol. 4, Analysis of Operators*, Acad. Press, New York, 1982.

[13] Eshkabilov Yu.Kh. Efimov's effect for a 3-particle model discrete Schrodinger operator. *Theor. Math. Phys.*, 2010, **164** (1), P. 896-–904.

[14] Eshkabilov Yu.Kh. On a discrete "three-particle" Schrodinger operator in the Hubbard model. *Theor. Math. Phys.*, 2006, **149** (2), P. 1497–1511.